

What is the spacetime of *physically realizable* spherical collapse?

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(Dated: August 31, 2002)

We argue that a particular spacetime, a spherically symmetric spacetime with hyper-surface orthogonal, radial, homothetic Killing vector, is a physically meaningful spacetime that describes the problem of spherical gravitational collapse in its full “physical” generality.

PACS numbers: 04.20.-q, 04.20.Cv

Keywords: Gravitational collapse - spherical symmetry - radial homothety - physically realizable

I. INTRODUCTION

Any sufficiently sparsely distributed ordinary “neutral” matter is dusty, that is, collision-less and pressureless. Further, emission of radiation and, hence, radiation is not expected in such dust.

In Newtonian gravity, as well as in general relativity, we can then study gravitational collapse from this dusty “initial” state of matter.

Under the action of its self-gravity, dust matter collapses. Then, self-gravity leads to mass or energy-flux in some preferential direction, the radial direction for a spherical spacetime.

But, this is not the flux of radiation. Therefore, there is no mass-flux in the rest frame of collapsing dusty matter, but it is present for other observers in the spacetime.

Next stage of collapse is reached when particles of dust begin to collide with each other. Negligible amount of radiation, but existing nonetheless, is expected from whatever atomic excitations or from whatever free electrons get created in atomic collisions in such matter. Therefore, dusty matter evolves into matter with pressure and radiation, both *simultaneously* non-vanishing.

The energy-flux can no longer be removed by going to the rest frame of matter.

As far as Newtonian gravity and general relativity, both, are concerned, we may study the gravitational collapse beginning even as matter with non-vanishing pressure and radiation.

At some stage, exothermic, thermonuclear reactions begin in matter with non-negligible pressure. With it, a star is born in the spacetime.

This is the manner of gravitational collapse of dusty matter leading to the birth of a star. Till the

exothermic thermonuclear reactions in the stellar core support the overlying stellar layers, such a stellar object is gravitationally stable.

But, the spacetime continues to be dynamic since radiation is present in it. The stellar object may also accrete matter from its surrounding while emitting radiation.

Once again, in Newtonian gravity and in general relativity, both, we may study the collapse of this “initial” stellar configuration of matter.

Now, as and when “heating” of the overlying stellar layers decreases due to changes in exothermic thermonuclear processes in the core of the star, the self-gravity of the stellar object leads to its gravitational contraction. These are, in general, very slow and involved processes.

Gravitational contraction leads to generation of pressure by compression and by the occurrence of exothermic thermonuclear reactions of heavier nuclei. The star may stabilize once more.

This chain, of gravitational contraction of star, followed by pressure increase, followed by subsequent stellar stabilization, continues as long as thermonuclear processes produce enough heat to support the overlying stellar layers.

The theory of the atomic nucleus shows that *exothermic* nuclear processes do not occur when Iron nucleus forms. With time, the rate of heat generation in iron-dominated-core becomes insufficient to support the overlying stellar layers which may then bounce off the iron-core resulting into a stellar explosion, a supernova.

Then, many, different such, stages of evolution are the results of physical processes that are unrelated to the phenomenon of gravitation. These are, for example, collisions of particles of matter,

electromagnetic and other forces between atomic or sub-atomic constituents of matter etc.

As an example, let some non-gravitational process, opposing collapse, result into pressure that does not appreciably rise in response to small contraction of the stellar matter. That is, pressure does not appreciably rise when gravitational field is increased by a small amount. Then, the collapse of a sufficiently massive object would not be halted by that particular non-gravitational process. Therefore, a *mass limit* is obtained in this situation. For example, electron degeneracy pressure leads to the Chandrasekhar limit [1].

Clearly, some of the non-gravitational processes determine the gravitational stability of physical objects. This is true in Newtonian gravity as well as in general relativity, both.

In general relativity, non-gravitational processes are included via the energy-momentum tensor for matter. Non-gravitational processes determine the relation of density and pressure of matter. The temporal evolution of matter is to be determined from such a relation, and from other physical relations, if any.

It is therefore that, by *physically realizable gravitational collapse*, we mean collapse that leads matter, step by step, through the above different “physical” stages of evolution.

Hence, the spacetime of “physically realizable” collapse of matter must be able to begin with any stage in the chain of evolution of matter under the action of its self-gravity. The temporal evolution from any “initial” data, any “physical” stage in question, is to be obtained from applicable non-gravitational properties of matter.

But, a supernova remnant, or a star that failed to explode, may be quite massive for its self-gravity to dominate over all conceivable competing reasons opposing it at various stages of further evolution. This may also happen as a result of mass-accretion taking the object in question over some mass-limit in operation. The collapse is, now, *unstoppable*. A spacetime singularity is expected to form in such unstoppable collapse.

Associated with studies of unstoppable collapse is the issue of whether the *physically realizable* gravitational collapse leads to a black hole or to a naked spacetime singularity. This is the issue of the Cosmic Censorship Hypothesis (CCH) [2].

Clearly, the answer to this very important question in general relativity can then be obtained only on the basis of the spacetime of the physically realizable gravitational collapse.

Now, we show below that, for spherical symmetry, a spacetime with hyper-surface orthogonal, radial, homothetic Killing vector provides “all” the above steps of evolution of matter.

II. SPACETIME OF PHYSICALLY REALIZABLE COLLAPSE

One radially homothetic spacetime has the following metric [3] in co-moving coordinates:

$$ds^2 = -y^2 dt^2 + \gamma^2 (y')^2 B^2 dr^2 + y^2 Y^2 d\Omega^2 \quad (1)$$

with $y = y(r)$, an overhead prime indicating a derivative with respect to r , $B \equiv B(t)$, $Y \equiv Y(t)$ and γ being a constant.

As can be easily verified, the metric (1) admits a spacelike Homothetic Killing Vector (HKV) of the form

$$X^a = (0, \frac{y}{\gamma y'}, 0, 0) \quad (2)$$

This is therefore the case of hyper-surface orthogonal, spacelike HKV.

Now, the spacetime of (1) is required, by definition, to be locally flat at all of its points including the center. The condition for elementary flatness at the center of (1) is

$$y'|_{r \sim 0} \approx 1/\gamma \quad (3)$$

This condition must be imposed on any $y(r)$. With this condition, (3), the HKV of metric (1) is, at the center, $y|_{r=0} \partial/\partial r$.

Now, we may use the function $y(r)$ in (1) as a new radial coordinate - the area coordinate - as long as $y' \neq 0$. However, the situation of $y' = 0$ represents a coordinate singularity that is similar to, for example, the one on the surface of a unit sphere where the analogue of y is $\sin \theta$ [4]. We can change the radial coordinate to suitable one before such a coordinate singularity is reached.

Singularities and degeneracies of metric (1)

The Ricci scalar for (1) is:

$$\begin{aligned} \mathcal{R} = & \frac{4\dot{Y}\dot{B}}{y^2 Y B} + \frac{2\ddot{B}}{y^2 B} - \frac{6}{y^2 \gamma^2 B^2} \\ & + \frac{2}{y^2 Y^2} + \frac{2\dot{Y}^2}{y^2 Y^2} + \frac{4\ddot{Y}}{y^2 Y} \end{aligned} \quad (4)$$

Then, there are two types of genuine curvature singularities of (1), namely, the first type for vanishing of temporal functions for some $t = t_s$ and, the second type for $y(r) = 0$ for some r .

The “physical” distance corresponding to the “coordinate” radial distance δr is $\ell = \gamma(y')B\delta r$. Then, collapsing matter forms the spacetime singularity in (1) when $\ell = 0$, ie, $B(t) = 0$ for it at

some $t = t_s$. Thus, the singularity of first type is a singular hyper-surface for (1).

The singularity of the second type is a singular sphere of coordinate radius r . The singular sphere reduces to a singular point for $r = 0$ that is the center of symmetry. Singularities of the second type constitute a part of the initial data, singular data, for the evolution.

The metric (1) has evident degeneracies when $y(r) = 0$, $y(r) \rightarrow \infty$ either on a degenerate sphere of coordinate radius r , for some “thick” shell or globally. Another degeneracy occurs for $y(r) = \text{constant}$ for some “thick” shell or globally.

In what follows, we shall assume that there is no singular initial-data and that there are no evidently degenerate situations for the metric (1).

Temporal evolution in (1)

The Einstein tensor for (1) is:

$$G_{tt} = \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} + \frac{\dot{Y}^2}{Y^2} + 2 \frac{\dot{B}\dot{Y}}{BY} \quad (5)$$

$$G_{rr} = \frac{\gamma^2 B^2 y'^2}{y^2} \left[-2 \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + \frac{3}{\gamma^2 B^2} - \frac{1}{Y^2} \right] \quad (6)$$

$$G_{\theta\theta} = -Y\ddot{Y} - Y^2 \frac{\ddot{B}}{B} - Y \frac{\dot{Y}\dot{B}}{B} + \frac{Y^2}{\gamma^2 B^2} \quad (7)$$

$$G_{\phi\phi} = \sin^2 \theta G_{\theta\theta} \quad (8)$$

$$G_{tr} = 2 \frac{\dot{B}y'}{By} \quad (9)$$

Clearly, the energy-flux depends on \dot{B} . Now, define the quantity

$$\sigma \equiv \sigma^1_1 = \sigma^2_2 = -\frac{1}{2}\sigma^3_3 = \frac{1}{3y} \left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B} \right) \quad (10)$$

Here, σ_{ab} represents the shear-tensor of the fluid and the shear-scalar is given by $\sqrt{6} \sigma$. Therefore, the spacetime of (1) is, in general, shearing and radiating, both.

With our assumptions of no singular and degenerate initial data, we then have a “cosmological” situation - continued spherical collapse of matter from the assumed “initial” state.

Now, for the co-moving observer with four-velocity $\mathbf{U} = \frac{1}{y} \frac{\partial}{\partial t}$, the *radial* velocity of the fluid is $V^r = \dot{Y}$ where an overhead dot denotes a time derivative. The co-moving observer is accelerating for (1) since $\dot{U}_a = U_{a;b} U^b$ is, in general, non-vanishing for $y' \neq 0$. The expansion is

$\Theta = \frac{1}{y} \left(\frac{\dot{B}}{B} + 2 \frac{\dot{Y}}{Y} \right)$ Now, we turn to steps of collapse of matter as outlined in §I.

Step I - Evolution of dust

Consider the collapse from “dusty” stage without radiation. Then, for vanishing energy-flux, $B \approx \text{constant} \equiv B_o$.

Then, the co-moving density, ρ , of dust is

$$\rho = \frac{1}{y^2} \left[\frac{\dot{Y}^2}{Y^2} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B_o^2} \right] \quad (11)$$

and the function $Y(t)$ is determined by the condition of vanishing of the isotropic pressure:

$$4Y\ddot{Y} + \dot{Y}^2 + 1 - \zeta Y^2 = 0 \quad (12)$$

Here, $\zeta = 5/\gamma^2 B_o^2$, a positive constant.

A solution of this equation is obtainable as

$$\frac{dY}{\sqrt{-1 + \zeta/5 Y^2 + c_o Y^{-1/2}}} = t - t_0 \quad (13)$$

where c_o is constant. Since \dot{Y} is the radial velocity of matter for the co-moving observer, we require that solution for which $\dot{Y} \rightarrow 0$ for $t \rightarrow -\infty$.

Step II - Evolution with pressure and radiation

Now, pressure and radiation, both, get *simultaneously* switched on in the spacetime of (1) when $B(t) \neq 0$. This is as per the expectation that dusty matter evolves to one with simultaneous occurrence of pressure and radiation, both.

Now, the co-moving density is

$$\rho = \frac{1}{y^2} \left[\frac{\dot{Y}^2}{Y^2} + 2 \frac{\dot{Y}\dot{B}}{YB} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} \right] \quad (14)$$

We also obtain

$$2 \frac{\ddot{Y}}{Y} + \frac{\ddot{B}}{B} = \frac{2}{\gamma^2 B^2} - \frac{y^2}{2} (\rho + 3p) \quad (15)$$

Then, from (15), the relation of pressure and density of matter is the required additional “physical” information. Also required is other relevant “physical” information to determine the radiation generation in the spacetime of (1).

To provide for the required information of “physical” nature is a non-trivial task in general relativity just as it is for Newtonian gravity. The details of these considerations are, of course, beyond the scope of this letter.

However, it is clear that the field equations determine only the temporal functions from the properties of matter in the spacetime of (1).

Moreover, it is also clear that matter will continue to pile up on such a star in a “cosmological setting” and, hence, such a star will always be taken over any mass-limit in operation at any stage of its evolution that will, ultimately, lead to the singular hyper-surface of the spacetime of (1).

The radial dependence of matter properties is “specified” as $1/y^2$ but the field equations of general relativity do not determine the metric function $y(r)$ in (1).

Therefore, the radial distribution of matter is *arbitrary* in terms of the co-moving radial coordinate r . This is the “maximal” *physical* freedom compatible with the assumption of spherical symmetry, we may note. Note, however, that the physical generality here is not to be taken to mean the “geometrical” generality.

III. ISSUE OF REGULARITY OF CENTER

A spherical spacetime admits an $SO(3)$ group of rotational symmetry. The orbits of the symmetry group are closed ones. The *center* of the spherically symmetric spacetime geometry is defined to be the “invariant point” of the $SO(3)$ group of rotations, as it must be.

Galilean invariance of the Newtonian equations implies that *every* observer observes the shrinkage of orbits of the rotation group to zero radius at the center of a spherically symmetric object.

Consequently, we may demand that the orbits of the rotation group also shrink to zero radius for a spherically symmetric spacetime. Such a spacetime is said to possess a *regular* center.

Now, for (1), $y(r)$ is the “area radius”. When $y|_{r=0} \neq 0$, the orbits of the rotation group $SO(3)$ do not shrink to zero radius at the center of the spacetime although the curvature invariants remain finite at the center. Also, when $y|_{r=0} = 0$, the orbits of the rotation group shrink to zero radius at the center but the curvature invariants blow up at the center, then.

It is well-known [5] that the center and the initial data for matter, both, are not *simultaneously* regular for a spherical spacetime with hyper-surface orthogonal HKV.

Therefore, the spacetime of (1) does not possess a *regular* center and *regular* matter data, simultaneously, and we may consider it to be “unphysical” even when its matter follows the expected “physical” evolution.

However, this issue requires careful analysis. It

is crucial to ask: who, which observer, is observing the orbits of the rotation group shrink to zero radius? This is important in general relativity though not in Newtonian gravity.

Consider the Schwarzschild spacetime. The asymptotic observer does not see any sphere, centered on the mass-point, shrink to any radius below $r = 2M$, the “infinite red-shift surface”.

Then, for the asymptotic observer, $r = 2M$ is the center of the spacetime! Thus, the “area radius” has this minimum value for the asymptotic observer.

It is no coincidence that this minimum value of the “area radius” as observed by the asymptotic observer is related to the gravitational mass or the Schwarzschild mass M .

Recall the well-know Newtonian theorem: “The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell’s matter were concentrated into a point at its center.”

The general relativistic manifestation of this Newtonian result is that the spacetime of a spherical body must possess non-vanishing central value for mass in that spacetime. For suitable observer, the orbits of the rotation group are not expected to shrink to zero radius at the center of a spherical spacetime with matter.

The co-moving observer of (1) is the “equivalent” of the asymptotic observer of the Schwarzschild spacetime in that it is also the “cosmological” observer for (1). Then, for this observer, orbits of the rotation group $SO(3)$ are not expected to shrink to zero “area radius” at the center of the spacetime of (1).

Therefore, the conflict of the “non-regularity” of the center and the “physical” evolution of matter in (1) can be resolved with this observation.

Now, let us call a star for which the orbits of the rotation group do not shrink to zero radius at its center a *strange* star, in contrast to the “standard” spherical star for which the center is regular. Then, it is clear from the above that *all* the spherical stars embedded in a cosmological surrounding are expected to be *strange* stars in the sense described above.

IV. CONCLUDING REMARKS

Many spacetimes of spherically symmetric nature are known [6]. For example, the “original” Schwarzschild [7], the “standard” Schwarzschild (or, Hilbert-Droste, [7]), the Vaidya, the Tolman-Bondi class, the Friedmann-Lemaître-Robertson-Walker spacetimes. Some of these known exam-

ples of spherically symmetric spacetime geometries contain no matter, only radiation dust, only matter dust, etc.

To construct the spacetime of a “physically realizable” spherical collapse of matter, we therefore need to “match” different, appropriate, such spacetime geometries. Of course, this is a non-trivial and, mostly, very difficult task.

But, these different spacetime geometries are, clearly, different choices of $y(r)$ in (1). For example, consider collapse from initial dusty matter with vacuum “exterior”. Then, $y(r)$ is infinite in the exterior.

The spacetime of (1) then provides the “final” spacetime that may be obtained after matching

many different spacetimes with appropriate physical conditions of matter. It is, therefore, the spacetime of *physically realizable collapse* of spherically symmetric matter.

We have, therefore, studied the shear-free collapse in [8] and the collapse with shear and energy flux in [9]. We have also obtained source free electromagnetic fields using the Hertz-Debye formalism [10] in [9]. In these studies, some general conclusions as well as the explicit forms for temporal metric functions have been presented for a simple equation of state of the barotropic form $p = \alpha\rho$ where α is a constant. We have also studied, in [11], the phenomenon of Hawking radiation in a radially homothetic spacetime of the metric (1).

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